Approximate Solutions of Seventh-Order Boundary Value Problems using Variational Iteration Algorithm via Chebyshev-Hermite Polynomials

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ABSTRACT

In this paper, the seventh-order boundary value problems (BVPs) are solved numerically in this study by utilizing the variational iteration approach and Chebyshev-Hermite polynomials. For the aforementioned boundary value problems, the suggested method translates to Chebyshev-Hermite polynomials, which are then employed as a basis function for the estimation. Furthermore, the results obtained demonstrate the effectiveness and dependability of the suggested strategy with some numerical examples using Maple 18 software.

(Keywords: seventh-order, boundary value problems, BVP, Chebyshev-Hermite polynomials, Probabilist's Hermite polynomials)

INTRODUCTION

Consider the seventh order BVP of the form:

$$v^7(\gamma) = G(\gamma, v(\gamma)), \alpha \leq \gamma \leq \beta \quad \textbf{(1)}$$

With boundary conditions:

$$v(\alpha) = \emptyset_0,\ v'(\alpha) = \emptyset_1,\ v''(\alpha) = \emptyset_2,\ v'''(\alpha) = \emptyset_3 \quad \textbf{(2)}$$

$$v(\beta) = \emptyset_4,\ v'(\beta) = \emptyset_5, v''(\beta) = \emptyset_6$$

A few of these problems have applications in science and technology including viscoelastic flow, heat transfer, fluid flow, etc. Numerous numerical techniques have been used over time to solve this type of problem. Solution of seventh-order BVPs Using canonical polynomials was presented in Abanum and Oke (2019) using a modified VIM. Also, a fifth order BVP employing He's polynomials is presented in Noor and Mohyud-din (2008).

Variational Iteration Method (VIM) tenth and ninth-order BVPs is with Modified Variational Iteration Method (MVIM) was also presented in (Tauseef and Yildirim, 2010). A unique technique for solving fifth order BVPs was presented by Noor and Mohyud-din (2010). For the solution of ninth order BVPs, Reddy (2016) employed the Quintic B-splines approach. For a BVP involving eight orders. Noor and Mohyud-din (2007), solve a seventh BVP utilizing He's polynomials.

Siddiqi and Iftikhar (2015) used the VIM for BVP. For the numerical solutions of generalized Nth order BVPs, Njoseh and Mamadu (2016) devised the power series approximation method. Additionally, the tau-collocation approximation approach was used by Mamadu and Njoseh (2016) to solve first and second order ordinary differential equations.

Fifth-order and other higher-order BVPs can be resolved using the Adomian decomposition method, which Noor and Mohyud-din (2007) and Noor and Mohyud-din, (2010), developed, as well as the VIM. Seventh-order BVPs were resolved by Siddiqi and Iftikhar (2013), using the variation of parameters method.

The power series approximation method was recently provided as a broad solution to this issue by Njoseh and Mamadu (2016). Additionally, Mamadu and Njoseh (2016) presented tau-collocation approximation method to solve first and second ODEs.

Using a sixth degree B-spline, Caglar, *et al.* (1999) also consider a numerical solution to the fifth order BVPs. The octal spline method was utilized by Akram and Siddiqi (2012) to arrive at a numerical solution to seventh order boundary value issues. Seventh order BVPs were solved by Viswanadham and Reddy (2013) using the Petrov-Galerkin Method with Quintic B-splines as Basis Functions and Septic B-splines as Weight Functions.

Other approaches can be found in Agarwal (1986), Richards and Sarma (1994), Siddiqi and Iftikhar (2013), Oyedepo, *et al*. (2019), Folasade, *et al.* (2021), Uwaheren, *et al.* (2021), Christie, *et al.* (2022), and Uwaheren, *et al.* (2022). The proposed methods work well, and the results are reliable and comforting. The solution is then presented in an infinite succession, leading to the result.

RESEARCH METHODS

The Standard VIM

Consider the differential equation below:

$$Lv + Nv - g(\gamma) = 0. \quad \textbf{(3)}$$

Where *L, N, and* $g(\gamma)$ represent the linear, nonlinear operator, and inhomogeneous terms, respectively. The correction functional as follows according to [26-27].

$$v_{m+1} = v_m(\gamma) + \int_0^{\gamma} \lambda(t)\big(Lv_m(t) + N\widetilde{v_m(t)} - g(t)\big)dt \quad \textbf{(4)}$$

$$\lim_{m\to\infty} v_m = v$$

The Lagrange Multiplier can be defined as follows:

$$(-1)^m \frac{1}{(m-1)!}$$

where *m* is the highest order of the differential equation.

Chebyshev-Hermite Polynomials

The Chebyshev-Hermite polynomials also referred to as "Probabilist's Hermite polynomials" is given by:

$$H_{e_n}(\gamma) = (-1)^n e^{\frac{\gamma^2}{2}} \frac{d^n}{d\gamma^n} e^{-\frac{\gamma^2}{2}}$$

Hence, the first few Chebyshev-Hermite Polynomials are given below:

$$H_{e_0}(\gamma) = 1,$$

$$H_{e_1}(\gamma) = \gamma,$$

$$H_{e_2}(\gamma) = \gamma^2 - 1, \quad \textbf{(5)}$$

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Modified Variational Iteration Algorithm using Chebyshev-Hermite Polynomials (MVIAC-HP)

Using **Equations (3)** and **(4)**, we assume an approximate solution of the form:

$$v_{m,N-1}(\gamma) = \sum_{m=0}^{N-1} a_{m,N-1} H_{e_{m,N-1}}(\gamma) \quad \textbf{(6)}$$

Where $H_{e_{m,N-1}}(\gamma)$ are Chebyshev-Hermite polynomials, $a_{m,N-1}$ are constants to be determined, and N the degree of approximant. Hence, we obtain the following iterative method:

$$v_{m+1,N-1}(\gamma) = \sum_{m=0}^{N-1} a_{m,N-1} H_{e_{m,N-1}}(\gamma) + \int_0^{\gamma} \lambda(t)\left(L\sum_{m=0}^{N-1} a_{m,N-1} H_{e_{m,N-1}}(t) + N_l \sum_{m=0}^{N-1} a_{m,N-1} H_{e_{m,N-1}}(t)\right)dt$$

RESULTS AND DISCUSSION

Two examples will be solved in this section with the proposed method.

Example 1 (Godspower, *et al.*, 2019; Shahid and Muzammal, 2015; Kasi, *et al.*, 2016)

The above authors considers the following seventh order linear BVP:

$$v^{(7)} + v + e^{\gamma}(35 + 12\gamma + 2\gamma^2) = 0, \qquad 0 \leq \gamma \leq 1. \quad \textbf{(7)}$$

With boundary conditions:

$$v(0) = 0,\ v'(0) = 1,\ \ v''(0) = 0,\ \ v'''(0) = -3$$

$$v(1) = 0,\ \ v'(1) = -e,\ \ v''(1) = -4e \quad \textbf{(8)}$$

The exact solution for the problem is $v(\gamma) = \gamma(1-\gamma)e^{\gamma}$.

The correction functional for the boundary value problem (7) and (8) is given as:

$$v_{m+1} = v_m(\gamma) + \int_0^{\gamma} \lambda(t)\left(v^{(7)} - v + e^t(35 + 12t + 2t^2)\right)dt$$

Where $\lambda(t) = \frac{(-1)^7(t-\gamma)^6}{6!}$ is the Lagrange multiplier.

Applying the VIM algorithm using the Chebyshev-Hermite polynomials, we assume an approximate solution of the form:

$$v_{m,6}(\gamma) = \sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(\gamma)$$

Hence, we get the following iterative formula:

$$v_{m+1,N-1}(\gamma) = \sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(\gamma) + \int_0^{\gamma} \frac{(-1)^7(t-\gamma)^6}{6!}\left(\frac{d^7}{dt^7}\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) - \left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) + e^t(35 + 12t + 2t^2)\right)dt$$

$$v_{m+1,N-1}(\gamma) = a_{0,6}H_{e_{0,6}}(\gamma) + a_{1,6}H_{e_{1,6}}(\gamma) + a_{2,6}H_{e_{2,6}}(\gamma) + a_{3,6}H_{e_{3,6}}(\gamma) + a_{4,6}H_{e_{4,6}}(\gamma) + a_{5,6}H_{e_{5,6}}(\gamma) + a_{6,6}H_{e_{6,6}}(\gamma) + \int_0^{\gamma} \frac{(-1)^7(t-\gamma)^6}{6!}\left(\frac{d^7}{dt^7}\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) - \left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) + e^t(35 + 12t + 2t^2)\right)dt$$

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From **Equation (5)**, iterating and applying the boundary conditions **Equation (8),** we obtain the unknown constants:

$a_{0,6} =$ -1.5000044, $a_{1,6} = -2.3749932$, $a_{2,6} = -3.5000129$, $a_{3,6} = -1.7499955$

$a_{4,6} = -0.83333737, \; a_{5,6} = -0.12499955, \; a_{6,6} = -0.033333589$

The series solution is expressed as

$$v(\gamma) = 2.250000000\ 10^{-7} + 1.00000005\gamma - 1.850000000\ 10^{-7}\gamma^2 - 0.50000000\gamma^3 - 0.333333535\gamma^4 - 0.12499955\gamma^5 - 0.033333589\gamma^6 - 0.006944444488\gamma^7 - 0.001190476192\gamma^8 - 0.0001736111101\gamma^9 - 0.00002204585538\gamma^{10} + O(\gamma^{11})$$

Table 1. Comparison of Numerical Results for Example 1.

x	Absolute Error by the proposed method	Absolute Error by (Kasi and Reddy, 2016)
0.1	2.281400e-07	1.415610e-07
0.2	2.274000e-07	6.407499e-07
0.3	2.225000e-07	2.920628e-06
0.4	2.137000e-07	4.410744e-06
0.5	2.011000e-07	6.735325e-06
0.6	1.854000e-07	6.407499e-06
0.7	1.667000e-07	3.665686e-06
0.8	1.449000e-07	3.278255e-07
0.9	1.194000e-07	1.430511e-06

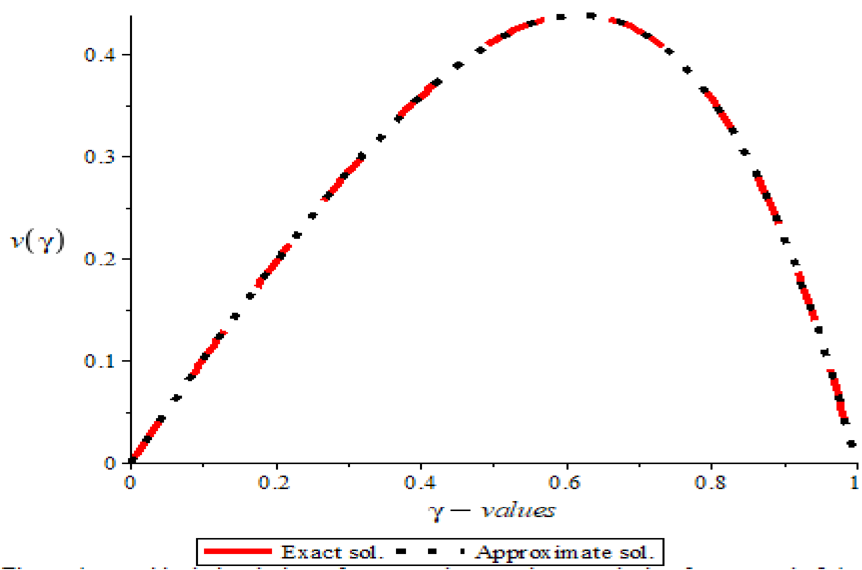

Figure 1. Graphical Simulation of Exact and Approximate Solution for Example 1.

Example 2 (Kasi, *et.al.*, 2015):

The above author considers the following seventh order linear BVP

$$v^{(7)} = vv' + T(\gamma), \qquad 0 \le \gamma \le 1 \tag{9}$$

With boundary conditions

$$\begin{aligned} &v(0) = 1,\ v'(0) = -2,\ v''(0) = 3,\ v'''(0) = -4 \\ &v(1) = 0,\ v'(1) = -e^{-1},\ v''(1) = 2e^{-1} \end{aligned} \tag{10}$$

Where: $T(\gamma) = e^{-2\gamma}(2 + e^{\gamma}(\gamma - 8) - 3\gamma + \gamma^2)$

The exact solution for the problem is $v(\gamma) = (1-\gamma)e^{-\gamma}$.

The correction functional for the boundary value problem (9) and (10) is given as:

$$v_{m+1} = v_m(\gamma) + \int_0^z \lambda(t)\left(v^{(7)} - vv' - e^{-2t}(2 + e^t(t-8) - 3t + t^2)\right)dt$$

Where $\lambda(t) = \frac{(-1)^7(t-\gamma)^6}{6!}$ is the Lagrange multiplier.

Applying the VIM algorithm using the Chebyshev-Hermite polynomials, we assume an approximate solution of the form:

$$v_{m,6}(\gamma) = \sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(\gamma)$$

Hence, we get the following iterative formula:

$$v_{m+1,N-1}(\gamma) = \sum_{m=0}^{N-1} a_{m,N-1} H_{e_{m,N-1}}(\gamma) + \int_0^{\gamma} \frac{(-1)^7(t-\gamma)^6}{6!} \left(\frac{d^7}{dt^7}\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) - \frac{d}{dt}\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right)\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) - e^{-2t}(2 + e^t(t-8) - 3t + t^2)\right)dt$$

$$v_{m+1,N-1}(\gamma) = a_{0,6}H_{e_{0,6}}(\gamma) + a_{1,6}H_{e_{1,6}}(\gamma) + a_{2,6}H_{e_{2,6}}(\gamma) + a_{3,6}H_{e_{3,6}}(\gamma) + a_{4,6}H_{e_{4,6}}(\gamma) + a_{5,6}H_{e_{5,6}}(\gamma) + a_{6,6}H_{e_{6,6}}(\gamma) + \int_0^{\gamma} \frac{(-1)^7(t-\gamma)^6}{6!} \left(\frac{d^7}{dt^7}\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) - \frac{d}{dt}\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right)\left(\sum_{m=0}^{6} a_{m,6} H_{e_{m,6}}(t)\right) - e^{-2t}(2 + e^t(t-8) - 3t + t^2)\right)dt$$

Hence: using **Equation (5)**, iterating and applying the boundary conditions **Equation (10)** the values of the unknown constants can be determined as follows:

$a_{0,6} = 3.27076079769953$, $a_{1,6} = -4.74988952964973$, $a_{2,6} = 3.18729227142704$
$a_{3,6} = -1.16659301592787$, $a_{4,6} = 0.354100716159602$, $a_{5,6} = -0.0499926357322523$

$a_{6,6} = 0.00971804504609165$

The series solution is given as:

$$v(\gamma) = 1.000000000 - 2.000000018\gamma + 1.500000001\gamma^2 - 0.6666666587\gamma^3 + 0.2083300405\gamma^4 - 0.0499926357322523\gamma^5 + 0.00971804504609165\gamma^6 - 0.001587301590\gamma^7 + 0.0002232142875\gamma^8 - 0.00002755731957\gamma^9 + 0.000003031283315\gamma^{10} + O(x^{11})$$

Table 2: Comparison of Numerical Results for Example 2.

x	Absolute Error by the proposed method	Absolute Error by (Kasi and Reddy, 2015)
0.1	2.000000e-09	8.344650e-07
0.2	6.700000e-09	6.377697e-06
0.3	1.690000e-08	1.281500e-05
0.4	3.250000e-08	1.686811e-05
0.5	4.870000e-08	1.746416e-05
0.6	5.770000e-08	1.436472e-05
0.7	5.310000e-08	8.881092e-06
0.8	3.472000e-08	3.449619e-06
0.9	1.537000e-08	3.911555e-07

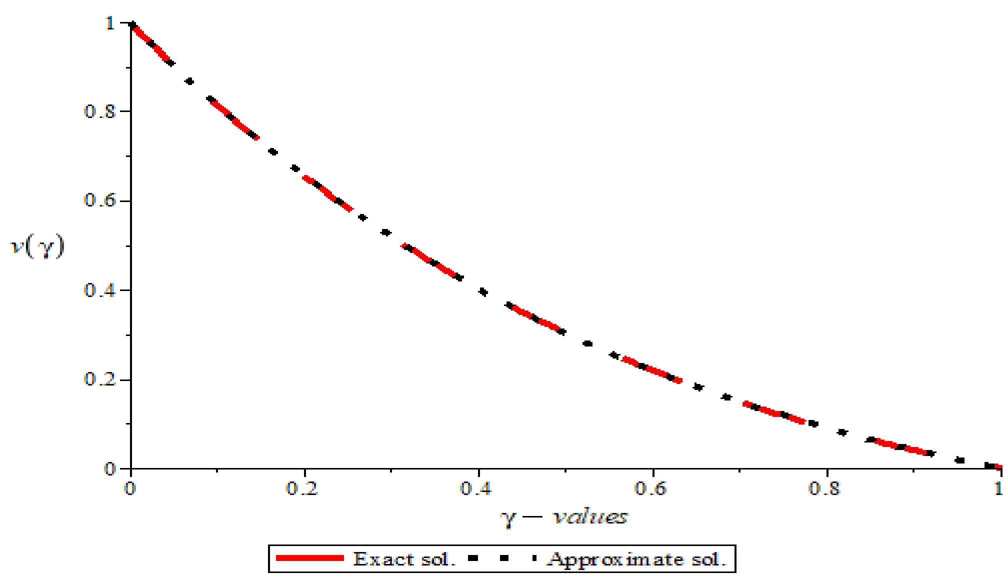

Figure 2: Graphical Simulation of Exact and Approximate Solution for Example 2.

CONCLUSIONS

In this paper, the variational iteration algorithm via Chebyshev-Hermite polynomials has been successfully applied to obtain the numerical solutions of seventh order boundary value problems. The method of solution involves Chebyshev Hermite polynomials mixed with variational iteration algorithm.

The method gives rapidly converging series solutions which occur in physical problem. From **Tables 1** and **2**, it is observed that the proposed method yields a better result when compared with methods in literature. Finally, the numerical results revealed that the present technique is a powerful mathematical tool for the solution of the class of problems put into consideration.

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